The Quantization of the Horava Gravity at the conformal kinetic point

LA PARTE Y EL TODO, AFUNALHUE 2017

Alvaro Restuccia, Universidad de Antofagasta

Outline

• Horava gravity, the main idea.
• Horava gravity at the conformal kinetic point.
• The action.
• The Hamiltonian formulation.
• The analysis of the constraints.
• The perturbation analysis around a Minkowski background.
• The propagator.
• The power counting renormalization.
• Conclusions
The main property of Horava’s proposal is to consider an anisotropic behaviour of space and time at high energies, the scaling being of the form

\[ t \sim [b]^z \]
\[ x \sim [b] \]

The scaling breaks the relativistic symmetry with the idea of recovering it, at least in an approximated sense, at low energies.
• The Horava’s proposal, following ideas from Lifshitz, for a renormalizable quantum theory of gravity gives rise to several models of gravity.

• All of them are in principle power counting renormalizable theories although a proof of it has to be explicitly given since the models are restricted by second class constraints and one has to show that the quantization procedure may be performed without the introduction of dangerous non-local terms and without introducing additional powers of momenta.
In the Horava proposal the space-time breaks into a spatial foliation parametrized by a preferred time. The foliation preserving diffeomorphisms of the three dimensional spatial leaves is the gauge symmetry of the theory.

The natural formulation of the theory is then in terms of the ADM variables \( g_{ij}, N, N_i \).
The main benefit would be to obtain a renormalizable theory of gravity, which is now possible since higher order spatial derivatives terms may be added to the potential without breaking the spatial diffeomorphism invariance of the action. In a relativistic theory together with the higher order spatial derivatives terms one has to include higher order time derivatives terms which introduces ghosts into the formulation and breaks the unitarity property of the theory.
At high energies the propagator is dominated by the higher spatial derivatives terms which improve the ultraviolet properties of the theory.

\[ \int \frac{d^3k}{\omega^2 - |\vec{k}|^2 - G(|\vec{k}|^2)} \]
The action

\[ S = \int dtd^3x \sqrt{g} N \left( \frac{1}{2\kappa} G^{ijkl} K_{ij} K_{kl} - \mathcal{V} \right) \]

\[ K_{ij} = \frac{1}{2N} (g_{ij} - 2\nabla (iN_j)), \]

\[ G^{ijkl} = \frac{1}{2} (g^{ik} g^{jl} + g^{il} g^{jk}) - \lambda g^{ij} g^{kl} \]

The coupling \( \lambda \) is dimensionless and the potential density is the most general combination of spatial derivatives of the Riemann tensor, the vector \( a_k = N_k/N \) and spatial derivatives of these objects transforming as scalars under spatial diffeomorphisms.
The final goal of obtaining a renormalizable theory may be achieved by considering a suitable scaling $z$ in order to have a dimensionless coupling on the action. For a four dimensional gravity theory the value of $z$ should be $z = 3$. In fact, 

The kinetic term has dimension $z + z$

\[ [dt \, d^D x] = -D - z \]

\[ [\kappa] = \frac{z - D}{2} \]
All interacting terms that contribute to the quadratic action in a 
perturbative approach around Minkowski space-time

\[-\mathcal{V}^{(z=1)} = \beta R + \alpha a_i a^i,\]

\[-\mathcal{V}^{(z=2)} = \alpha_1 R \nabla_i a^i + \alpha_2 \nabla_i a_j \nabla^i a^j + \beta_1 R_{ij} R^{ij} + \beta_2 R^2,\]

\[-\mathcal{V}^{(z=3)} = \alpha_3 \nabla^2 R \nabla_i a^i + \alpha_4 \nabla^2 a_i \nabla^2 a^i + \beta_3 \nabla_i R_{jk} \nabla^i R^{jk} + \beta_4 \nabla_i R \nabla^i R,\]
The conformal transformation which leaves the kinetic term invariant, when $\lambda = 1/3$, is

\[ g_{ij} \rightarrow e^{2\Omega} g_{ij}, \quad N \rightarrow e^{3\Omega} N, \]
\[ N_i \rightarrow e^{2\Omega} N_i, \quad \Omega = \Omega(t, \vec{x}), \]
The tensor $G^{ijkl}$ is not invertible for $\lambda = 1/3$ hence two different theories with different physical degrees of freedom arises from the two cases: $\lambda = 1/3$ and $\lambda \neq 1/3$. Most of the work has been done for $\lambda \neq 1/3$: In this case the theory has one additional physical degree of freedom compared to GR. In particular, the strong coupling problem has been extensively analyzed. The case we consider corresponds to $\lambda = 1/3$, the theory contains exactly the same number of physical degrees of freedom as General Relativity, they may be realized in terms of the transverse-traceless components of the three dimensional metric as in GR.
The $\pi = 0$ constraint emerges when the theory is formulated at the conformal point

\[
\frac{\pi^{ij}}{\sqrt{g}} = \frac{1}{2\kappa} G^{ijkl} K_{kl}
\]

\[
\lambda = 1/3 \quad g_{ii} G^{ijkl} = 0
\]
\[
\int d^3x \left( N \, g^{-1/2} \, \pi_{ij} \pi^{ij} + g^{1/2} \, NV + N_i H^i + \sigma \, \varphi + \omega \, \pi \right) + E_{ADM}
\]

\[H^i = -2 \, D_k \, \pi^{ik} + \varphi \, \partial^i N\]

\(\sigma, \omega\) are Lagrange multipliers,
\(\varphi = 0\) and \(\pi = 0\) primary constraints,
\(\varphi\) is the conjugate momentum associated to \(N\).
\[ E_{\text{ADM}} = \int d\Sigma_i \left( \partial_k g_{ik} - \partial_i g_{kk} \right) \]

It has to be included in the hamiltonian in order to obtain the equations of motion compatible with the boundary conditions from the most general variations of the hamiltonian. [Regge,Teitelboim].

The preservation of primary constraint yields: \( H = 0, \ C = 0 \)

\[ H = g^{-1/2} \pi_{ij} \pi^{ij} - g^{1/2} R + \alpha g^{1/2} (2 D_i a^i + a_i a^i) + \omega g^{1/2} C_{ij} C^{ij} \]

\[ C = 3/2 N g^{-1/2} \pi_{ij} \pi^{ij} + \frac{1}{2} N g^{1/2} R - N g^{1/2} (2 D_i a^i + (2 - \alpha/2) a_i a^i) \]
The conservation of these new constraints provides two elliptic partial differential equations for the lagrange multipliers $\sigma$, $\omega$. The Dirac algorithm ends up at this stage.

The constraint $H_i = 0$ is a first class constraint while $\varphi = 0$, $\pi = 0$, $H = 0$ and $C = 0$ are second class ones.

The primary constraint $\pi = 0$ is a particular property of the Horava theory formulated for $\lambda = 1/3$. This value is protected from quantum corrections.
The *physical degrees of freedom* are obtained by eliminating from the first class constraint the longitudinal part of the momentum and of the three dimensional metric by a gauge fixing, $N$ and one component of the three dimensional metric together with the trace part of the momentum are eliminated from the second class constraints. One is left with the transverse traceless components of the metric and its conjugate momentum as the physical degrees of freedom of the theory as in GR. Moreover, if we consider a perturbative analysis around a Minkowski background, which is an exact solution of the field equations, and we consider the low energy Hamiltonian at the quadratic level, it is exactly the Hamiltonian of GR.
Perturbations around Minkowski space-time

\[ g_{ij} = \delta_{ij} + \epsilon h_{ij}, \quad \pi^{ij} = \epsilon p_{ij}, \quad N = 1 + \epsilon n. \]

\[ h_{ij} = h_{ij}^{TT} + \frac{1}{2} (\delta_{ij} - \partial_{ij} \partial^{-2}) h^T + \partial_i h_j^L, \]

\[ \partial_i p_{ij} = 0, \quad \partial_i h_{ij} = 0, \]

\[ p^T = 0. \]

\[ \{ h_{ij}^{TT}, p_{ij}^{TT}, h^T, n \} \]
The $H = 0$ and $C = 0$ constraints

\[ \phi = \begin{pmatrix} h^T \\ n \end{pmatrix}, \quad M = \begin{pmatrix} D_1 & D_2 \\ D_2 & D_3 \end{pmatrix}, \]

\[ M\phi = 0 \]

\[ D_1 \equiv \frac{1}{8}((3\beta_3 + 8\beta_4)\partial^6 - (3\beta_1 + 8\beta_2)\partial^4 + \beta\partial^2), \]

\[ D_2 \equiv \frac{1}{2}(\alpha_3\partial^6 + \alpha_1\partial^4 + \beta\partial^2), \quad D_3 \equiv \alpha_4\partial^6 - \alpha_2\partial^4 + \alpha\partial^2. \]
Two decoupled elliptic equations for \( n \) and \( h^T \):

\( L \) is a sixth-order polynomial in the laplacian. It can always be factorized in terms of monomials.

\[
L \phi = 0, \quad L \equiv D_1 D_3 - D_2^2.
\]

\[
L = K(\partial^2 - z_1)P^{(5)}(\partial^2)
\]

\[
(\partial^2 - z_1)\psi = 0
\]

\[
(\partial^2 - z_1)^{-1} \quad (-\infty, 0]
\]
The reduced Hamiltonian: 
the solution of the constraints under the assumption on the coupling constants:

\[ \beta(2\beta - \alpha) \neq 0. \]

\[ h_T = n = 0. \]

\[ H_{\text{RED}} = \int d^3x \left( 2\kappa p_{i;j}^{TT} p_{i;j}^{TT} + \frac{1}{4} h_{i;j}^{TT} \nabla h_{i;j}^{TT} \right), \]

\[ \nabla = -\beta \partial^2 - \beta_1 \partial^4 + \beta_3 \partial^6. \]
The perturbative (linearized) behavior of the effective action at large distances is exactly as linearized General Relativity. The field equations for $h^{TT}$ are wave equations. Consequently, the perturbative formulation of the theory around Minkowski space-time, at large distances, propagates gravitational waves exactly as linearized General Relativity. However, the nonperturbative dynamics of both theories is different, even at the lowest $z=1$ order in the Horava formulation.
The positivity of the Hamiltonian imposes restrictions on the coupling constants.

• We consider positivity of the potential

\[ V = -\beta \partial^2 - \beta_1 \partial^4 + \beta_3 \partial^6 \]

• Positivity at low energies imply \( \beta > 0 \) and at high energies \( \beta_3 < 0 \). \( \beta=0 \) is excluded in order to have the correct physical degrees of freedom, \( \beta_3=0 \) is excluded in order to have a genuine \( Z=3 \) theory.

• If in addition \( \beta_1 \leq 0 \) then the potential is positive.

• If \( \beta_1 > 0 \)
The Fourier transform of the operator $V$

\[ \tilde{V}(k^2) = |\beta_3| k^2 (k^2 - |z_+|)(k^2 - |z_-|) \]

\[ V = \beta_3 \partial^2 (\partial^2 - z_+)(\partial^2 - z_-), \]

\[ z_\pm = \frac{1}{2\beta_3} \left( \beta_1 \pm \sqrt{\beta_1^2 + 4\beta \beta_3} \right). \]
The restrictions on the coupling constants in order to have:

- the same linearized behaviour as GR at low energies
- The Z = 3 order (power counting renormalizability)
- positivity of the Hamiltonian

Are:

\[ \alpha \neq 2\beta, \quad \beta > 0, \quad \beta_3 < 0, \quad \beta_1 \leq 2\sqrt{\beta|\beta_3|}. \]
The propagator of the physical degrees of freedom

\[ \langle h_{ij}^{TT} h_{kl}^{TT} \rangle = \frac{P_{ijkl}^{TT}}{\omega^2 / 2\kappa - \beta k^2 + \beta_1 k^4 + \beta_3 k^6}, \]

\[ P_{ijkl}^{TT} \equiv \frac{1}{\sqrt{2}}(\theta_{ik}\theta_{jl} + \theta_{il}\theta_{jk} - \theta_{ij}\theta_{kl}), \quad \theta_{ij} \equiv \delta_{ij} - \frac{k_i k_j}{k^2} \]
We then notice that at high energies the anisotropic sixth order term dominates over the relativistic second order one, the propagator acquires additional powers of the spatial momentum compared to the relativistic propagator. This additional powers of momentum improve the UV properties of the theory. At low energies the theory flows to the $z = 1$ theory which has the same propagator as GR.

The relativistic scaling for the propagator is then restored at low energies.
Solving the constraints at higher order in perturbation theory

\[ 2\epsilon (\mathbb{D}_2 h^T + \mathbb{D}_3 n) = \frac{\epsilon^2}{4} \left[ -8k p_{ij}^T p_{ij}^T + \beta_1 \partial^2 h_{ij}^{TT} \partial^2 h_{ij}^{TT} \\
+ \beta_3 \partial^2 \partial_i h_{jk}^{TT} \partial^2 \partial_i h_{jk}^{TT} \\
+ (\beta + \alpha_1 \partial^2 + \alpha_3 \partial^4)(4h_{ij}^{TT} \partial^2 h_{ij}^{TT} \\
+ 3\partial_i h_{jk}^{TT} \partial_i h_{jk}^{TT} - 2\partial_i h_{jk}^{TT} \partial_k h_{ij}^{TT} \right], \]

If \((1/8)(\alpha_4(3\beta_3 + 8\beta_4) - 2\alpha_3^2)\) is different from zero the solutions for \(h^T\) and \(n\) do not introduce additional powers of momenta into the vertices.
Power counting renormalizability

For each loop
\[ \int d\omega d^d k \rightarrow \Lambda^{d+z}, \]

for each propagator \( \Lambda^{-2z} \)

For each vertex (at most) \( \Lambda^{2z} \)

\[ D \leq (d + z)L + 2z(V - I) \]
\[ = (d - z)L + 2z(L + V - I) \]
\[ D \leq 2z. \]
Conclusions

• Propagation of gravitational waves on a Minkowski background.
• Unitarity and power counting renormalizability.
• Path integral formulation.
• Spherically symmetric solutions: wormholes.

• Wave zone in Horava gravity at the kinetic conformal point.
• Renormalization.